We can take what we learned in the last sections and apply it to even more differential equations!

Differential Equations

Class Notes

The Superposition Principle and Undetermined Coefficients Revisited (Section 4.5)

We will find general solutions and solve initial value problems involving nonhomogeneous diff. eqs.. We will start with our fundamental theorem.

## Theorem 3: The Superposition Principle:

Let  $y_1$  be a solution to the diff. eq.  $ay'' + by' + cy = f_1(t)$ . Let  $y_2$  be a solution to the diff. eq.  $ay'' + by' + cy = f_2(t)$ . Then for the diff. eq.  $ay'' + by' + cy = k_1 f_1(t) + k_2 f_2(t)$  (where  $k_1$  and  $k_2$  are any constants), we know that  $k_1 y_1 + k_2 y_2$  will be a solution.

The book provides the tiniest proof for this.

expl 1; Through the miracle that is a math book, we know that

$$y_1(t) = (\frac{1}{4})\sin(2t)$$
 is a solution to the diff. eq.  $y'' + 2y' + 4y = \cos(2t)$ 

Also, we are given that  $y_2(t) = \frac{t}{4} - \frac{1}{8}$  is a solution to the diff. eq. y'' + 2y' + 4y = t.

Use the superposition principle to find solutions to the following.

(a.) 
$$y'' + 2y' + 4y = 2t - 3\cos(2t)$$

$$y = 2(\frac{t}{4} - \frac{1}{8}) - 3(\frac{1}{4} \sin(2t))$$

$$y = \frac{t}{2} - \frac{1}{4} - \frac{3}{4} \sin(2t)$$

(b.) 
$$y'' + 2y' + 4y = 5t$$
  
 $y'' = \frac{5}{4} + \frac{5}{8}$ 

We can also use the superposition principle to find a general solution to ay'' + by' + cy = f(t) by using the fact that a particular solution to it is  $y_p$  and that the general solution to ay'' + by' + cy = 0 is  $c_1y_1 + c_2y_2$  (where  $y_1$  and  $y_2$  are solutions from a previous section). Our next theorem spells this out for initial value problems.



Theorem 4: Existence and Uniqueness: Nonhomogeneous Case:

For any real numbers a ( $a \neq 0$ ), b, c,  $t_0$ ,  $Y_0$ , and  $Y_1$ , suppose  $y_p(t)$  is a particular solution to ay'' + by' + cy = f(t) in an interval I containing  $t_0$  and that  $y_1(t)$  and  $y_2(t)$  are linearly independent solutions to the associated homogeneous equation ay'' + by' + cy = 0 in I. Then there exists a unique solution in I to the initial value problem ay'' + by' + cy = f(t),  $y(t_0) = Y_0$ ,  $y'(t_0) = Y_1$ . This solution is (drum roll please)

 $y(t) = y_p(t) + c_1y_1(t) + c_2y_2(t)$  using the appropriate choice of the constants  $c_1$  and  $c_2$ 

expl 2: Given the nonhomogeneous equation with a particular solution below, find a general

solution.  $y'' + 5y' + 6y = 6x^2 + 10x + 2 + 12e^x$ ,  $y_p(x) = e^x + x^2$ 

Solve auxiliary equation  $r^2 + 5r + 6 = 0$  to determine a general solution to y'' + 5y' + 6y = 0.

r2+5r+6=0 (r+3)(r+2) = 0

r=-3,-2 -> from 4,2! 40 = C1e-3x + C2e

Superposition principle:  $y = y_9 + y_P$   $y = C_1e^{-3x} + C_2e^{-2x} + e^x + x^2$   $y' = -3c_1e^{-3x} - 2c_2e^{-2x} + e^x + 2x$  $y'' = 9c_1e^{-3x} + 4c_2e^{-2x} + e^{x} + 2$ +54 +64 = 9cie-3x + 4c2e-2x +ex +2

-15c1e-3x-10c2e-2x+5ex +10x  $+6C_1e^{-3x} + 6C_2e^{-2x} + 6e^x + 6x^2 = 6x^2 + 10x + 2$ 

C1, C2 unless given initial values)

Next, we will see how to solve the type of equation where the nonhomogeneity f(t) follows a specific form, similar to what we saw in a previous section.





To find a particular solution to the diff. eq.  $ay'' + by' + cy \neq P_m(t)e^{rt}$  where  $P_m(t)$  is a polynomial of degree m, use the form  $y_p(t) = t^s \left( A_m t^m + ... + A_1 t + A_0 \right) e^{rt}$ . We use the following values for s.



- i.) Use s = 0 if r is not a root of the associated auxiliary equation.
- ii.) Use s = 1 if r is a simple root of the associated auxiliary equation.
- iii.) Use s = 2 if r is a double root of the associated auxiliary equation.

To find a particular solution to the diff. eq.

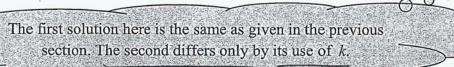
 $ay'' + by' + cy = P_m(t)e^{\alpha t} \cos \beta t + Q_n(t)e^{\alpha t} \sin \beta t$  where  $\beta$  is non-zero, and where  $P_m(t)$  is a polynomial of degree m and  $Q_n(t)$  is a polynomial of degree n, use the form

$$y_p(t) = t^s (A_k t^k + ... + A_1 t + A_0) e^{\alpha t} \cos \beta t + t^s (B_k t^k + ... + B_1 t + B_0) e^{\alpha t} \sin \beta t$$

Here, k is the larger of m and n. We use the following values for s.

iv.) Use s = 0 if  $\alpha + i\beta$  is *not* a root of the associated auxiliary equation.

v.) Use s = 1 if  $\alpha + i\beta$  is a *simple* root of the associated auxiliary equation.

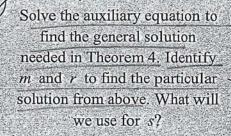




expl 3: Find a general solution to the diff. eq. below.

$$y''-2y'-3y(3t^2-5)e1)$$
  $3t^2-5=P_1$ 

 $r=3,-1 \rightarrow y_{g} = C_{1}e^{3t} + C_{2}e^{-t}$   $y_{p} = t^{2}(A_{2}t^{2} + A_{1}t + A_{0})e^{9}$   $y_{p} = A_{2}t^{2} + A_{1}t + A_{0}$ 





(extra room for work)

10 = A2t + A, t + A0 And Az, A, Ao: y = 2A2++ A1

Use the diff, eq. to find the values for coefficients  $A_2$ ,  $A_1$ , and Ao. Once found, substitute back in and form general solution as

 $y(t) = y_p(t) + c_1 y_1(t) + c_2 y_2(t)$ .

Be you careful! The diff. eq. is hard but the algebra is harder!

origean: y"-zy'-3y = 3t2-5

 $2A_2 - 2(2A_2t + A_1) - 3(A_2t^2 + A_1t + A_0) = 3t^2 - 5$ 

 $2A_2 - 4A_2t - 2A_1 - 3A_2t^2 - 3A_1t - 3A_0 = 3t^2 - 0t - 5$ 

-3A2=3 -4A2-3A1=0

2A2-2A1-3A0=-5

4-3A1=0 -3A=-4 (Ae=-1

A1 = 4/3

-2-8 - 3A0=-5

= -3A0=-3

(Ao = 1)q

Soln: y = C1e3t + C2e-t - t2 + 3t + 9 You can

always check your solution.



## **Derivative Calculator:**

There are times when our particular solution  $y_p(t)$  will be nasty and finding its derivatives will be onerous. If that happens, feel free to use a derivative calculator online. Here are some I found.

https://www.derivative-calculator.net/

https://www.wolframalpha.com/calculators/derivative-calculator

But don't overuse them so your brain does not mush up.

## **Initial Value Problems:**

expl 4: Find the solution to this initial value problem.

y'' = (6t) y(0) = 3, y'(0) = -1

amp egn:

r=0 (doubte root)

(section 4.2) > yg = Ciect + C2 t. es

A double root for the aux. eqn. will set up a general solution (with  $c_1$  and  $c_2$ ) for the related homogeneous eqn.. Then use method from page 3 to write particular solution with coefficients.

(page 3) f(t) = 6t, eot

r=0 isa

Pm(t) double rest

(m=1) of aux equ

Use diff. eq. to determine values of coefficients. Finish by finding  $c_1$  and c<sub>2</sub> using the initial values.

= t2 (A, t + A0) &

4" = 6Ait + 2Ao

y=C1+C2++t3>

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## What if the nonhomogeneity is not one of the types mentioned?

We can break up the right-side nonhomogeneity f(t) if needed and use the Superposition Principle to cobble together a particular solution to more complicated diff. eqs..

We may also need to use such gems as the trig identity  $\cos 3\theta = 4\cos^3 \theta - 3\cos \theta$ .

$$4\cos^3\theta = \cos 3\theta + 3\cos \theta = \frac{1}{4}\cos(3\theta)$$

expl 5: Determine the form of a particular solution for the diff. eq. Do not solve.

This 
$$f(t)$$
 does *not* fit into our

 $x''-x'-2x = e^{t} \frac{\cos t - t^{2} + \cos^{3} t}{2}$   $x''-x'-2x = e^{t} \frac{\cos t - t^{2} + \cos^{3} t}{2}$   $+ \frac{1}{4} \cos(3t) + \frac{3}{4} \cos t$ 

This f(t) does *not* fit into our rules. But its individual terms do, after some rearranging.

XPI = A et cost + Betsuit

M=0 X=1 → S=0 B=1  $\frac{r^2 - r - 2 = 0}{(r + 1)(r - 2)} = 0$ 

Xp2 = (A2t2+A1t+A0) 2001

(XP3 = C cos (3t) + Dsin (3t)

$$M = 0$$
  $\beta = 3$   $\alpha = 0$   $\beta = 3$ 

Sch:

$$Xp = Ae^{t} c \times t + Be^{t} s in t + A_{2}t^{2} + A_{1}t + A_{0}t + C c \times (3t) + D s in (3t) + E c \times t + F s in t$$