We can take what we learned in the last sections and apply it to even more differential equations!

Differential Equations
Class Notes
The Superposition Principle and Undetermined Coefficients Revisited (Section 4.5)

We will find general solutions and solve initial value problems involving nonhomogeneous diff. eqs.. We will start with our fundamental theorem.

Theorem 3: The Superposition Principle:
Let y_1 be a solution to the diff. eq. $ay''+by'+cy=f_1(t)$. Let y_2 be a solution to the diff. eq. $ay''+by'+cy=f_2(t)$. Then for the diff. eq. $ay''+by'+cy=k_1f_1(t)+k_2f_2(t)$ (where k_1 and k_2 are any constants), we know that $k_1 y_1 + k_2 y_2$ will be a solution.

The book provides the tiniest proof for this.

expl 1: Through the miracle that is a math book, we know that $y_1(t)=\left(\frac{1}{4}\right)\sin(2t)$ is a solution to the diff. eq. $y''+2y'+4y=\cos(2t)$.
Also, we are given that $y_2(t)=\frac{t}{4}-\frac{1}{8}$ is a solution to the diff. eq. $y''+2y'+4y=t$.
Use the superposition principle to find solutions to the following.

a.) $y''+2y'+4y=2t-3\cos(2t)$

$$y = 2\left(\frac{t}{4}-\frac{1}{8}\right) - 3\left(\frac{1}{4}\sin(2t)\right)$$

$$\boxed{y = \frac{t}{2}-\frac{1}{4}-\frac{3}{4}\sin(2t)}$$

b.) $y''+2y'+4y=5t$

$$y = \frac{5}{4}t - \frac{5}{8}$$

We can also use the superposition principle to find a **general** solution to $ay''+by'+cy=f(t)$ by using the fact that a particular solution to it is y_p and that the general solution to $ay''+by'+cy=0$ is $c_1y_1+c_2y_2$ (where y_1 and y_2 are solutions from a previous section). Our next theorem spells this out for initial value problems.

Theorem 4: Existence and Uniqueness: Nonhomogeneous Case:
For any real numbers a $(a \neq 0)$, b, c, t_0, Y_0, and Y_1, suppose $y_p(t)$ is a particular solution to $ay'' + by' + cy = f(t)$ in an interval I containing t_0 and that $y_1(t)$ and $y_2(t)$ are linearly independent solutions to the associated homogeneous equation $ay'' + by' + cy = 0$ in I.
Then there exists a unique solution in I to the initial value problem
$ay'' + by' + cy = f(t)$, $y(t_0) = Y_0$, $y'(t_0) = Y_1$. This solution is (drum roll please)
$y(t) = y_p(t) + c_1 y_1(t) + c_2 y_2(t)$ using the appropriate choice of the constants c_1 and c_2.

expl 2: Given the nonhomogeneous equation with a particular solution below, find a general solution.

$y'' + 5y' + 6y = \underbrace{6x^2 + 10x + 2 + 12e^x}_{\text{"nasty" } f(x)}$, $y_p(x) = e^x + x^2$

Solve auxiliary equation $r^2 + 5r + 6 = 0$ to determine a general solution to $y'' + 5y' + 6y = 0$.

$r^2 + 5r + 6 = 0$
$(r+3)(r+2) = 0$
$r = -3, -2$ → from 4.2: $y_g = c_1 e^{-3x} + c_2 e^{-2x}$

Superposition principle: $y = y_g + y_p$

$$\boxed{y = c_1 e^{-3x} + c_2 e^{-2x} + e^x + x^2}$$

Check:
$$y' = -3c_1 e^{-3x} - 2c_2 e^{-2x} + e^x + 2x$$
$$y'' = 9c_1 e^{-3x} + 4c_2 e^{-2x} + e^x + 2$$
$$\begin{aligned} y'' + 5y' + 6y = {} & 9c_1 e^{-3x} + 4c_2 e^{-2x} + e^x + 2 \\ & -15c_1 e^{-3x} - 10c_2 e^{-2x} + 5e^x + 10x \\ & +6c_1 e^{-3x} + 6c_2 e^{-2x} + 6e^x + 6x^2 = 6x^2 + 10x + 2 + 12e^x \end{aligned}$$
✓

(Can't find c_1, c_2 unless given initial values)

Next, we will see how to solve the type of equation where the nonhomogeneity $f(t)$ follows a specific form, similar to what we saw in a previous section.

Method of Undetermined Coefficients for Certain Nonhomogeneities Involving Polynomials (Revisited):

To find a particular solution to the diff. eq. $ay'' + by' + cy = P_m(t)e^{rt}$ where $P_m(t)$ is a polynomial of degree m, use the form $y_p(t) = t^s\left(A_m t^m + \ldots + A_1 t + A_0\right)e^{rt}$. We use the following values for s.

i.) Use $s = 0$ if r is *not* a root of the associated auxiliary equation.
ii.) Use $s = 1$ if r is a *simple* root of the associated auxiliary equation.
iii.) Use $s = 2$ if r is a *double* root of the associated auxiliary equation.

To find a particular solution to the diff. eq.
$ay'' + by' + cy = P_m(t)e^{\alpha t}\cos\beta t + Q_n(t)e^{\alpha t}\sin\beta t$ where β is non-zero, and where $P_m(t)$ is a polynomial of degree m and $Q_n(t)$ is a polynomial of degree n, use the form

$$y_p(t) = t^s\left(A_k t^k + \ldots + A_1 t + A_0\right)e^{\alpha t}\cos\beta t + t^s\left(B_k t^k + \ldots + B_1 t + B_0\right)e^{\alpha t}\sin\beta t\,.$$

Here, k is the larger of m and n. We use the following values for s.

iv.) Use $s = 0$ if $\alpha + i\beta$ is *not* a root of the associated auxiliary equation.
v.) Use $s = 1$ if $\alpha + i\beta$ is a *simple* root of the associated auxiliary equation.

The first solution here is the same as given in the previous section. The second differs only by its use of k.

expl 3: Find a ~~general solution~~ to the diff. eq. below.

$y'' - 2y' - 3y = (3t^2 - 5)\cdot 1 \rightarrow 3t^2 - 5 = P_m(t)$

$s = 0$

aux eqn:

$r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0$

$r = 3, -1 \rightarrow y_g = C_1 e^{3t} + C_2 e^{-t}$

$y_p = \cancel{t^{0}}(A_2 t^2 + A_1 t + A_0)\cancel{e^{0t}}$

$y_p = A_2 t^2 + A_1 t + A_0$

Solve the auxiliary equation to find the general solution needed in Theorem 4. Identify m and r to find the particular solution from above. What will we use for s?

(extra room for work)

Use the diff. eq. to find the values for coefficients A_2, A_1, and A_0. Once found, substitute back in and form general solution as $y(t) = y_p(t) + c_1 y_1(t) + c_2 y_2(t)$.

Be you careful! The diff. eq. is hard but the algebra is harder!

$$y_p = A_2 t^2 + A_1 t + A_0$$

find A_2, A_1, A_0:

$$y' = 2A_2 t + A_1$$

$$y'' = 2A_2$$

orig eqn: $y'' - 2y' - 3y = 3t^2 - 5$

$$2A_2 - 2(2A_2 t + A_1) - 3(A_2 t^2 + A_1 t + A_0) = 3t^2 - 5$$

$$2A_2 - 4A_2 t - 2A_1 - 3A_2 t^2 - 3A_1 t - 3A_0 = 3t^2 - 0t - 5$$

$-3A_2 = 3$

$A_2 = -1$

$-4A_2 - 3A_1 = 0$

$4 - 3A_1 = 0$

$-3A_1 = -4$

$A_1 = 4/3$

$2A_2 - 2A_1 - 3A_0 = -5$

$-2 - \frac{8}{3} - 3A_0 = -5$

$\frac{-8}{3} - 3A_0 = -3$

$A_0 = 1/9$

Soln:

$$\boxed{y = c_1 e^{3t} + c_2 e^{-t} \underbrace{- t^2 + \frac{4}{3}t + \frac{1}{9}}_{y_p}}$$

You can always check your solution.

4.5

Derivative Calculator:

There are times when our particular solution $y_p(t)$ will be nasty and finding its derivatives will be onerous. If that happens, feel free to use a derivative calculator online. Here are some I found.

https://www.derivative-calculator.net/

https://www.wolframalpha.com/calculators/derivative-calculator

But *don't* overuse them so your brain does not mush up.

Initial Value Problems:

expl 4: Find the solution to this initial value problem.

$y'' = 6t, \quad y(0) = 3, \quad y'(0) = -1$

A *double* root for the aux. eqn. will set up a general solution (with c_1 and c_2) for the related homogeneous eqn.. Then use method from page 3 to write particular solution with coefficients.

aux eqn:

$r^2 = 0$

$r = 0$ (double root)

(section 4.2) $\rightarrow y_g = C_1 \cancel{e^{0t}}^{1} + C_2 t \cdot \cancel{e^{0t}}^{1}$

$y_g = C_1 + C_2 t$

(page 3) $\rightarrow f(t) = 6t \cdot e^{0t}$

$6t \rightarrow P_m(t)$ $(m=1)$

$e^{0t} \rightarrow$ $r=0$ is a double root of aux eqn so $s=2$.

Use diff. eq. to determine values of coefficients. Finish by finding c_1 and c_2 using the initial values.

$y_p = t^2(A_1 t + A_0)\cancel{e^{0t}}^{1}$

$y_p = A_1 t^3 + A_0 t^2$

$y' = 3A_1 t^2 + 2A_0 t$

$y'' = 6A_1 t + 2A_0$

$y'' = 6t$

$6A_1 t + 2A_0 = 6t + 0$

$6A_1 = 6 \qquad 2A_0 = 0$

$A_1 = 1 \qquad A_0 = 0$

$\rightarrow y_p = t^3$

$y = C_1 + C_2 t + t^3$

Find C_1, C_2

$y(0)=3$: $3 = C_1$

$y'(0) = -1$: $y' = C_2 + 3t^2$

$-1 = C_2$

soln

$$\boxed{y = t^3 - t + 3}$$

What if the nonhomogeneity is *not* one of the types mentioned?

We can break up the right-side nonhomogeneity $f(t)$ if needed and use the Superposition Principle to cobble together a particular solution to more complicated diff. eqs..

We may also need to use such gems as the trig identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

$$4\cos^3\theta = \cos 3\theta + 3\cos\theta \rightarrow \cos^3\theta = \frac{1}{4}\cos(3\theta) + \frac{3}{4}\cos(\theta)$$

expl 5: Determine the form of a particular solution for the diff. eq. Do *not* solve.

$x'' - x' - 2x = e^t\cos t - t^2 + \cos^3 t$

> This $f(t)$ does *not* fit into our rules. But its individual terms do, after some rearranging.

$$x'' - x' - 2x = e^t\cos t - t^2 + \frac{1}{4}\cos(3t) + \frac{3}{4}\cos t$$

aux eqn:

$r^2 - r - 2 = 0$

$(r+1)(r-2) = 0$

$r = -1, 2$

For $f(t) = e^t\cos t$: (section 4.4) $m=0$, $\alpha = 1$, $\beta = 1 \rightarrow s=0$

$$x_{p1} = Ae^t\cos t + Be^t\sin t$$

For $f(t) = -t^2$: (section 4.4 or 4.5) $m=2$, $r=0 \rightarrow s=0$

$$x_{p2} = (A_2t^2 + A_1t + A_0)\cancel{e^{0t}}^{\,1}$$

For $f(t) = \frac{1}{4}\cos(3t)$: (section 4.4) $m=0$, $\alpha=0$, $\beta=3 \rightarrow s=0$

$$x_{p3} = C\cos(3t) + D\sin(3t)$$

For $f(t) = \frac{3}{4}\cos t$: (section 4.4) $m=0$, $\alpha=0$, $\beta=1 \rightarrow s=0$

$$x_{p4} = E\cos t + F\sin t$$

Soln:

$$x_p = Ae^t\cos t + Be^t\sin t + A_2t^2 + A_1t + A_0 + C\cos(3t) + D\sin(3t) + E\cos t + F\sin t$$